ISOMETRIC EQUIVALENCE OF ISOMETRIES ON H^p

Joseph A. Cima *and* **Warren R. Wogen**

Abstract.We consider a natural notion of equivalence for bounded linear operators on H^p, for $p \neq 2$. We determine which isometries of finite codimension are equivalent. For these isometries , we classify those which have the Crownover property.

1. Introduction

If A and B are bounded linear operators on a Hilbert space, then A and B are unitarily equivalent if $B = UAU^*$ for some unitary operator U. One can then view A and B as abstractly the same operator. In the general Banach space setting one can replace unitary equivalence by using onto isometries of the Banach space being considered. So if X is a Banach space and $\mathbb{B}(X)$ are the bounded linear operators on X, then we will say that A and B in $\mathbb{B}(X)$ are isometrically equivalent if $B = UAU^{-1}$ for some onto isometry U in $\mathbb{B}(X)$. In this case we write $A \approx B$. The utility of this notation will of course depend on specific properties of the space X and its onto isometries. The Banach spaces considered in this note are the classical Hardy spaces H^p, for $p \neq 2$. The onto isometries of H^p have been determined (see Theorem A) , and we will classify some familiar operators on H^p up to isometric equivalence. This work is motivated by some questions of J. Jamison. In particular he asked which isometries on H^p are equivalent to the shift (see Corollary 1).

2. Preliminaries

In this paper we consider the Banach spaces H^p of the unit disc D , for $1 \leq p < \infty, p \neq 2$. Recall that H^p consists of the analytic functions f on D for which

$$Sup_{0<r<<1} \int_T |f(r\zeta)|^p dm(\zeta)$$

is finite , dm the usual Lebesgue measure on the unit circle T (See Duren2]).

Definition 1. $\mathbb{I}(H^p)$ *will denote the onto isometries of* H^p , $p \neq 2$.

$\mathbb{I}(H^p)$ is a group under the usual operator multiplication. The description of $\mathbb{I}(H^p)$ for $p = 1$ is due to de Leew,et.al. [4] , and for $1 < p < \infty$, $p \neq 2$, to Forelli [6] and is given in Theorem A below.

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Definition 2. *Let $\mathbb{A}$ be the collection of holomorphic automorphisms of the unit disc D. That is,*

$$\mathbb{A} = \{\phi(z) = \frac{\lambda(z-a)}{1-\overline{a}z}; \quad a \in D, \quad |\lambda| = 1\}$$

$\mathbb{A}$ is a group under composition with identity e, where $e(z) = z$. Following [8], we write ϕ_n for the n-fold composition of ϕ with itself. In addition, we denote the (compositional) inverse of ϕ by ϕ_{-1}. Note that $(\phi_n)_{-1}$, the inverse of ϕ_n, is just $(\phi_{-1})_n$, the n-fold composition of ϕ_{-1} with itself, which we denote by ϕ_{-n}.

Definition 3. *For ϕ, and ψ in $\mathbb{A}$ we say that ϕ is conjugate to ψ if there is an $\eta \in \mathbb{A}$ with $\phi = \eta \circ \psi \circ \eta_{-1}$. The conjugacy class of ϕ is denoted $\mathbb{C}(\phi)$ and so $\mathbb{C}(\phi) = \{\eta \circ \phi \circ \eta_{-1} \ : \eta \in \mathbb{A}.\}$*

The following proposition is well known and we include it for notational purposes.

Proposition 1. *Suppose (b_k) is a sequence of functions in $\mathbb{A}$ and that $b_k(a_k) = 0 \ \forall \ k \geq 1$.*

(i) If $\prod_{k=1}^{\infty} b_k(z)$ is a convergent Blaschke product, then (a_k) is a Blaschke sequence. That is, $\sum_{k=1}^{\infty}(1-|a_k|) < \infty$.

(ii) Conversely, if (a_k) is a Blaschke sequence then there exists $(\lambda_k)_{k=1}^{\infty} \in T$ so that $\prod_{k=1}^{\infty} \lambda_k b_k(z)$ converges.

3. Isometric Equivalence

A theorem of Forelli [6] describes all isometries of H^p onto H^p. For $\phi \in \mathbb{A}$ and $d \in T$ Forelli showed that the map

$$f \mapsto d(\phi'(z))^{1/p} f \circ \phi$$

is in $\mathbb{I}(H^p)$, and that all onto isometries have this form. Theorem A below is a restatement of this result.

If

$$\phi(z) = \frac{\lambda(a-z)}{1-\overline{a}z},$$

then

$$\phi'(z) = \frac{\lambda(1-|a|^2)}{(1-\overline{a}z)^2},$$

so that the choice of a branch of the pth root function that will make $(\phi'(z))^{1/p}$ analytic will depend on λ. It is useful to set our notation so that we always use the principal branch given by $(r\exp(\imath\theta))^{1/p} = r^{1/p}\exp(\imath\theta/p)$, $-\pi < \theta < \pi$, $r > 0$. Now $(\overline{\lambda}\phi'(z)) = \frac{(1-|a|^2)}{(1-\overline{a}z)^2}$ has positive real part so $(\overline{\lambda}\phi'(z))^{1/p}$ is analytic on D.

Let $C_\phi f = f \circ \phi$ denote composition by ϕ and for $F \in H^\infty, M_F$ denotes multiplication by F. Finally, let

$$U_\phi = M_{((\overline{\lambda})\phi')^{1/p}} C_\phi.$$

Forelli's result can be stated as

Theorem *A*. $\mathbb{I}(H^p) = \{\rho U_\phi \, : \phi \in \mathbb{A} \, , |\rho| = 1\}$

From this result we see that $\mathbb{I}(H^p)$ is determined by $\mathbb{A}$. We will examine the relation between the group structure of $\mathbb{A}$ and $\mathbb{I}(H^p)$.

We note that for $p = 2$, the operators of the form ρU_ϕ in Theorem A are of course unitary operators on H^2 which are tied to the analytic structure of H^2. These unitaries are a small subgroup of the full unitary group on H^2.

Lemma 1. *Let ϕ, and $\psi \in \mathbb{A}$. Then*

a)$U_\phi U_\psi = \rho U_{\psi\circ\phi}$ for some $\rho \in T$, which depends on ϕ and ψ.

b)$U_\phi^{-1} = U_{\phi_{-1}}$.

Proof. Suppose that $\phi(z) = \frac{\lambda_1(z-a_1)}{1-\overline{a_1}z}$ and $\psi(z) = \frac{\lambda_2(z-a_2)}{1-\overline{a_2}z}$. Then

$$\psi \circ \phi(z) = \frac{\lambda_3(z-a_3)}{1-\overline{a_3}z}$$

for some $\lambda_3 \in T, a_3 \in D$. Note that $C_\psi C_\phi = C_{\phi\circ\psi}$. Also if $F \in H^\infty$, then $C_\phi M_F = M_{F\circ\phi} C_\phi$. Thus

$$U_\phi U_\psi = M_{(\lambda_1 \ \phi')^{1/p}} C_\phi M_{(\lambda_2 \ \psi')^{1/p}} C_{\psi\circ\phi} = (\overline{\lambda_1}\phi')^{1/p}(\overline{\lambda_2}\psi' \circ \phi)^{1/p} C_{\psi\circ\phi}.$$

But

$$U_{\psi\circ\phi} = (\overline{\lambda_3}(\psi\circ\phi)')^{1/p} C_{\psi\circ\phi} = (\overline{\lambda_3}(\psi'\circ\phi)\phi')^{1/p} C_{\psi\circ\phi}$$

So one sees that $U_{\psi\circ\phi}$ is a unimodular multiple of $U_\phi U_\psi$, and a) is proven.

For part b) we recall that

$$\phi_{-1}(z) = \frac{\overline{\lambda_1}(z + \lambda_1 a_1)}{(1 + \overline{\lambda_1 a_1} z)}.$$

Take $\psi = \phi_{-1}$ in the last proof. Thus

$$U_\phi U_{\phi_{-1}} = (\overline{\lambda_1}\,\phi')^{1/p}(\lambda_1 \phi'_{-1} \circ \phi)^{1/p} C_{\phi_{-1}\circ\phi} = I \quad \#$$

Remark. The value of the constant ρ in Lemma 1 a) will not be needed in our work, but it can of course be explicitly computed. For ϕ and ψ as in the proof of Lemma 1, one can show that $\rho = \exp i\theta$, where $\theta = arg(1 + \overline{\lambda_1 a_1} a_2)^{2/p}$.

We now describe all $S \in \mathbb{I}(H^p)$ which are isometrically equivalent to a fixed $U_\phi \in \mathbb{I}(H^p)$.

Proposition 2. *$S \approx U_\phi \Leftrightarrow$ there exists $\eta \in \mathbb{A}$ and $\rho \in T$ so that*

$$S = \rho U_{\eta\circ\phi\circ\eta_{-1}}.$$

Proof: $S \approx U_\phi \Leftrightarrow$ there exists $\eta \in \mathbb{A}$ so that

$$U_{\eta_{-1}} U_\phi U_\eta = S.$$

But

$$U_{\eta_{-1}} U_\phi U_\eta = U_{\eta_{-1}}(\rho_1 U_{\eta\circ\phi}) = \rho_2\rho_1 U_{\eta\circ\phi\circ\eta_{-1}}$$

where $\rho_1, \rho_2 \in T$ as in Lemma 1a. #

So Proposition 2 states that

$$\widetilde{\psi} \in \mathbb{C}(\phi) \Leftrightarrow U_\phi = \rho U_{\widetilde{\psi}}$$

for some $\rho \in T$.

We focus on the isometries of H^p into H^p. The most familiar example is the shift, M_z, on H^p. The range of M_z is zH^p so is of codimension one. For example, which $S \in \mathbb{B}(H^2)$ satisfy $S \approx M_z$? We will in fact classify all finite codimension isometries up to isometric equivalence. We give the details of our results for the case for codimension one isometries and the codimension n case follows similarly.

Note that M_z has the additional property that

$$\bigcap_{n=1}^{\infty} (M_z)^n H^p = (0).$$

Definition 4. *A codimension one isometry S on H^p is called Crownover (see [2] , [7]) , if $\bigcap_{n=1}^{\infty} S^n H^p = (0)$.*

We will also classify such isometries up to isometric equivalence.

4. Finite Codimensional Isometries

We will now state Forelli's theorem [6, Theorem 1] describing all isometries of H^p, $p \neq 2$.

Theorem B. S is an isometry of H^p, $1 \leq p < \infty$, $p \neq 2$ iff $S = Ff(\phi)$ for some ϕ inner and an $F \in H^p$ which is related to ϕ.

The precise relationship between F and ϕ can be found in [6] and is not needed in the work that follows. We will provide a simpler description of the isometries of finite codimension.

Lemma 2. *Suppose that $T = M_F C_\phi$ as in Theorem B and that the inner function ϕ is not in $\mathbb{A}$. Then the T has infinite codimension.*

Proof: We will modify the proof of [1 , Lemma 3.6]] . Let $K_b(z) = \frac{1}{1-\overline{b}z}$.

Since ϕ is an open map which is not univalent, we can choose sequences $(a_n), (b_n)$ in D so that $\phi(a_n) = \phi(b_n) = c_n \ \forall\, n$. F is not the zero function so we can also assusme $F(a_n) \neq 0$ and $F(b_n) \neq 0 \ \forall\, n$. Let $g_n = \overline{F(b_n)}K_{a_n} - \overline{F(a_n)}K_{b_n}$. Since the kernels are linearly independent the functions g_n are linearly independent. The g_n are in H^∞ and so induce linear functionals Λ_n on H^p which are linearly independent and satisfy

$$\Lambda_n(Tf) = \Lambda_n(Ff \circ \phi) = F(b_n)F(a_n)c_n - F(b_n)F(a_n)c_n = 0,$$

for all $f \in H^p$. Hence,

$$\bigcap Ker(\Lambda_j) \supset T(H^p).$$

Thus $\{g_n\}$ is a linearly independent set whose span intersects TH^p only at (0). This implies T has infinite codimension. #

Hence we need only consider isometries of the form $M_F C_\phi$, where $\phi \in \mathbb{A}$ and $F \in H^p$. If $\phi(z) = \frac{\lambda(z-c)}{1-\overline{c}z}$, then $M_F C_\phi = M_{\frac{F}{(\overline{\lambda}\,\phi')^{1/p}}} U_\phi$. Since $U_\phi \in \mathbb{I}(H^p)$, it follows that $M_{\frac{F}{(\overline{\lambda}\,\phi')^{1/p}}}$ must be isometric. This means that $M_{\frac{F}{(\overline{\lambda}\,\phi')^{1/p}}}$ is an inner function, which we label as Ψ.

Clearly $M_\Psi U_\phi$ has the codimension of M_Ψ. The codimension is $n < \infty \Leftrightarrow \Psi$ is an n-fold Blaschke product. In this case we write $\Psi \in \mathbb{A}_n$. In particular $\mathbb{A}_1 = \mathbb{A}$.

We have shown that the set of isometries of codimension n is given by

$$\mathbb{I}_n(H^p) = \{M_\Psi U_\phi \ : \ \phi \in \mathbb{A}\,, \Psi \ \in \mathbb{A}_n\ \}.$$

In most of what follows, we focus on the isometries

$$\mathbb{I}_1(H^p) = \{M_\psi U_\phi \ : \phi, \psi \in \mathbb{A}\}$$

of codimension one.

Theorem 1. *Let $S_1 = M_\psi U_\phi \in \mathbb{I}_1(H^p)$. If $S_2 \in \mathbb{I}_1(H^p)$, then $S_2 \approx S_1 \Leftrightarrow \exists \eta \in \mathbb{A}$ and $\rho \in T$ so that $S_2 = M_{\rho\psi\circ\eta} U_{\eta_{-1}\circ\phi\circ\eta}$.*

Proof: $S_2 \approx S_1 \Leftrightarrow \exists \eta \in \mathbb{A}$ so that $U_{\eta_{-1}} S_1 U_\eta = S_2$. But

$$\begin{aligned} U_\eta S_1 U_{\eta-1} = U_\eta M_\psi U_\phi U_{\eta_{-1}} &= M_{\psi\circ\eta} U_\eta(\rho_1 U_{\eta_{-1}\circ\phi}) = M_{\psi\circ\eta}\rho_1(U_\eta U_{\eta_{-1}\circ\phi}) \\ &= M_{\psi\circ\eta}\rho_1\rho_2 U_{\eta_{-1}\circ\phi\circ\eta} = \rho M_{\psi\circ\eta} U_{\eta_{-1}\circ\phi\circ\eta}. \end{aligned}$$

Here ρ_1 and ρ_2 are the unimodular constants that arise in Lemma 1 a, and $\rho = \rho_1\rho_2$. #

With $e(z) = z$ note that if $\psi \in \mathbb{A}$, then $M_\psi = M_\psi U_e$ has codimension one .

Corollary 1. *If $\psi \in \mathbb{A}$ and $S \in \mathbb{I}_1(H^p)$, then $S = M_{\widetilde{\psi}}$ for some $\widetilde{\psi} \in \mathbb{A}$.*

Proof: $S \approx M_\phi \Leftrightarrow \exists \eta \in \mathbb{A}$ so that

$$S = U_\eta M_\psi U_{\eta_{-1}} = M_{\psi\circ\eta} U_\eta U_{\eta_{-1}} = M_{\psi\circ\eta}.$$

Finally, note that $\{\psi \circ \eta \ : \ \eta \in \mathbb{A}\} = \mathbb{A}$ #

We remark that the above result shows that $S \approx M_z \Leftrightarrow S = M_\psi$ for some $\psi \in \mathbb{A}$. We now generalize the last corollary. Fix $\phi \in \mathbb{A}$ and consider when $M_\psi U_\phi \approx M_{\widetilde{\psi}} U_\phi$. Corollary 1 settles the question if $\phi = e$.

So suppose $\eta \in \mathbb{A}$ and that $U_\eta (M_\psi U_\phi) U_{\eta_{-1}} = M_{\widetilde{\psi}} U_\phi$. The left side simplifies to

$$M_{\psi \circ \eta} U_\eta U_\phi U_{\eta_{-1}} = \rho M_{\psi \circ \eta} U_{\eta_{-1} \circ \phi \circ \eta},$$

and with the notation $\widetilde{\psi} = \rho \psi \circ \eta$ and $\phi = \eta_{-1} \circ \phi \circ \eta$ we have our equality

$$U_\eta (M_\psi U_\phi) U_{\eta_{-1}} = M_{\widetilde{\psi}} U_\phi.$$

It follows that $\phi \circ \eta = \eta \circ \phi$, so ϕ and η commute. Thus we have

Corollary 2. *$M_\psi U_\phi \approx M_{\widetilde{\psi}} U_\phi \Leftrightarrow \widetilde{\psi} = \rho \psi \circ \eta$ for some $\eta \in \mathbb{A}$ with η commuting with ϕ and $\rho \in T$ satisfying*

$$U_\eta U_\phi U_{\eta_{-1}} = \rho U_{\eta_{-1} \circ \phi \circ \eta}.$$

Remark: We will discuss in Section 7 the classification of the automorphisms commuting with a fixed $\phi \in \mathbb{A}$.

Recall that $\forall\ \psi \in \mathbb{A}, M_\psi$ is a Crownover shift. That is

$$\bigcap_{n=1}^{\infty} (M_\psi)^n H^p = \bigcap_{n=1}^{\infty} (M_{(\psi)^n})) H^p = (0).$$

Given a $S = M_\psi U_\phi \in \mathbb{I}_1(H^p)$ when is S Crownover? Now

$$S^2 = (M_\psi U_\phi)((M_\psi U_\phi) = M_\psi M_{\psi \circ \phi} U_\phi U_\phi = \rho M_\psi M_{\psi \circ \phi} U_{\phi_2},$$

for some $\rho \in T$. Iterating we have

$$S^n = (M_\psi U_\phi)^n = \rho M_\psi M_{\psi \circ \phi} \ldots . M_{\psi \circ \phi_{n-1}} U_{\phi_n},$$

where $\rho \in T$ depends on n.

Now U_{ϕ_n} is onto , so $S^n H^p = B_n H^p$, where B_n is the Blaschke product $\prod_{k=0}^{n-1} b_k$, where

$$b_k = \psi \circ \phi_k.$$

Note that b_k is merely the kth term of the sequence $(\psi \circ \phi_k)$ and does not represent the kth iterate of b.

It follows that $\bigcap_{n=1}^{\infty} S^n H^p = \bigcap_{n=1}^{\infty} B_n H^p$. If this intersection contains an $f \neq 0$ then each b_k is a factor of f so by Proposition 4 there is a Blaschke product of the form $B = \prod_{k=0}^{\infty} \lambda_k b_k$ such that $\bigcap_{n=1}^{\infty} S^n H^p = B H^p$. Thus the zeros of $(b_k)_{k=0}^{\infty}$ form a Blaschke sequence. The above discussion shows that

Theorem 2. *$M_\phi U_\phi$ is Crownover $\Leftrightarrow$ the sequence of zeros of $(\psi \circ \phi_k)_{k=0}^{\infty}$ is not a Blaschke sequence.*

We will elaborate on this result in the next section using the fixed point structure of ϕ.

At this time we maintain the terminology as above , assuming that $S = M_\psi U_\phi$ and that $B = \prod_{k=0}^{\infty} \lambda_k b_k$ is an infinite Blaschke product, with $\lambda_0 = 1$. Note that BH^p is an invariant subspace for S. We will show that $S|_{BH^p} \in \mathbb{I}(BH^P)$. First note that

$$M_\psi C_\phi B = M_\psi C_\phi \prod_0^\infty \lambda_k b_k = \psi \prod_0^\infty \lambda_k b_{k+1}$$

So if $g \in H^p$, then

$$SBg = M_\psi U_\phi Bg = BU_\phi g,$$

and $S|_{BH^p}$ is onto BH^p.

Lastly, we note that $S|_{BH^P}$ is isometrically equivalent to U_ϕ. Let $V : H^p \to BH^p$ be the isometry defined by

$$V_g = Bg \ , \ g \in H^p.$$

Then

$$g \in H^p \Rightarrow (S|_{BH^p})V_g = S(Bg) = BU_\phi g,$$

so that $S|_{BH^p} \approx U_\phi$.

Remark: We now consider the case as above but with $p = 2$. The Wold decomposition for the isometry S (see [9, Th.1.1]), is easy to exhibit. Namely, $H^2 = BH^2 \bigoplus (BH^2)^\perp$ is a direct sum of invariant subspaces of S. $S|_{BH^2}$ is unitary and is in fact unitarily equivalent to U_ϕ, while $S|_{(BH^2)^\perp}$ is a unilateral shift. If $\psi(z) = \frac{\mu(z-b)}{1-\bar{b}z}$, then span of the kernel K_b is a wandering subspace for the shift.

5. The Crownover Property

Each $\phi \in \mathbb{A}, \phi \neq e$, can be classified as elliptic, hyperbolic, or parabolic according to its fixed points in $\overline{D}$. See [1] or [8] for more detail.

Definition 5. *$\phi \in \mathbb{A}$, $\phi \neq e$, is elliptic if ϕ has a fixed point, say a, in D. Let $\mathbb{E}(a) = \{\psi \in \mathbb{A} \ : \ \psi(a) = a, \ \psi \neq e\}$ denote the set of all elliptic automorphisms of D that fix a.*

Choose $\eta \in \mathbb{A}$ with $\eta(a) = 0$ and note that $\eta \circ \mathbb{E}(a) \circ \eta_{-1}$ is the set of nontrivial rotations of D.

Definition 6. *ϕ is parabolic if it has only one fixed point , say w. In this case $w \in T$ and $\phi'(w) = 1$. Of course w is also the unique fixed point of ϕ_{-1}. w is attractive for ϕ (and for ϕ_{-1}). That is, for all $c \in D, \phi_n(c) \to w$ and $\phi_{-n}(c) \to w$.*

Definition 7. *$\phi \in \mathbb{A}$, $\phi \neq e$ is called hyperbolic if ϕ has two distinct fixed points, say w_1, and w_2, on T. In this case one of the fixed points , say w_1, is the attractive fixed point for ϕ. Also w_2 is the attractive fixed point for ϕ_{-1}. Further, $\phi'(w_1) < 1$ and $\phi'(w_2) > 1$.*

Let

$$\mathbb{H}(w_1\ ,\ w_2) = \{\psi \in \mathbb{A}\ , \phi \neq e\ : \psi(w_1) = w_1\ ,\ \psi(w_2) = w_2\}$$

be the collection of hyperbolic automorphisms that fix w_1 and w_2. If $w_1^{'}, w_2^{'}$ is another pair of distinct points on T and $\eta \in \mathbb{A}$ is chosen so that $\eta(w_1) = w_1^{'}, \eta(w_2) = w_2^{'}$, then $\mathbb{H}(w_1^{'}, w_2^{'}) = \eta \circ \mathbb{H}((w_1, w_2) \circ \eta_{-1}$. Thus all of these sets are conjugate.

As an example , take $w_1 = -1$, $w_2 = 1$. Then one can show $\mathbb{H}(-1, 1) = \{\psi_r\ ;\ -1 < r < 0\ or\ 0 < r < 1\}$, where $\psi_r(z) = \frac{z-r}{1-rz}$.

Proposition 3. *If ϕ is elliptic and $\psi \in \mathbb{A}$, then $M_\psi U_\phi$ is Crownover.*

Proof: Since ϕ is conjugate to a rotation , it is routine to check that the zeros of $\psi \circ \phi_n$ lie on a circle in D and hence can not be a Blaschke sequence. #

Proposition 4. *If $\psi \in \mathbb{A}$ and ϕ is hyperbolic , than $M_\psi U_\phi$ is not Crownover.*

Proof: It is easy to check that if ϕ is hyperbolic and $c \in D$ that $\sum(1 - |\phi_n(c)|) < \infty$. (See $[8, p85, \#6.)$ Suppose $\psi \circ \phi_n(a_n) = 0\ \forall\ n \geq 0$. Then $\phi_n(a_n) = \psi_{-1}(0)$, and $a_n = \phi_{-n} \circ \psi_{-1}(0)$. But ϕ_{-1} is also hyperbolic , so $\sum(1 - |\phi_{-1}(\psi_{-1}(0)|) = \sum(1 - |a_n|) < \infty$ and (a_n) is a Blaschke sequence. #

Our goal is to show that if $\phi, \psi \in \mathbb{A}$ with ϕ parabolic, then $M_\psi U_\phi$ is not Crownover. That is , the zeros of $(\psi \circ \phi_n)$ form a Blaschke sequence, just as in the case that ϕ is hyperbolic.

Definition 8. *For $w \in T$, let $\mathbb{P}(w)$ be the collection of parabolic automorphisms that fix w.*

It is easy to see that the sets $\mathbb{P}(w)$, $w \in T$, are conjugate to each other. So we first consider $\mathbb{P}(1)$.

A computation will show that

$$\phi(z) = \frac{\lambda(z-a)}{1 - \overline{a}z} \in \mathbb{P}(1) \Leftrightarrow \phi(1) = 1 = \phi'(1).$$

Solving for a and λ, we see that

$$|a - 1/2| = 1/2\ ,\ a \neq 0, 1$$

and that $\lambda = \frac{1-\overline{a}}{1-a}$. So $a - 1/2 = (c/2)$, where $c \in T, c \neq \pm 1$ and thus $\lambda = \frac{1-\overline{c}}{1-c} = \frac{-1}{c}$. Using these equalities we can write ϕ in the form

$$\phi(z) = \frac{1 + c - 2z}{2c - (1+c)z},$$

which we write as $\phi_c(z)$.

So we have

$$\mathbb{P}(1) = \{\phi_c(z) = \frac{1 + c - 2z}{2c - (1+c)z}\ \ :\ \ c \in T\ , c \neq \pm 1\}\ .$$

The functions $\phi_\imath$ and $\phi_{-\imath}$ play a special role in what follows.

Observe that if $\phi \in \mathbb{P}(1)$ and if $\psi \in \mathbb{A}$ with $\psi(1) = 1$, then

$$\psi \circ \phi \circ \psi_{-1} \in \mathbb{P}(1).$$

Here ψ could be hyperbolic. Our approach is to conjugate $\phi_\imath$ (or $\phi_{-\imath}$) by automorphisms $\psi_r \in \mathbb{H}(-1,1)$, discussed after Definition 7. We note that the inverse of ψ_r is ψ_{-r}.

Proposition 5. *Let* $\phi_c(z) = \frac{1+c-2z}{2c-(1+c)z}$ *where* $c \in T$, $c \neq \pm i$. *If* $\Im(c) > 0$, *then* $\exists\ r \in (-1,1)$, $r \neq 0$ *and* $\psi_r \in \mathbb{H}(-1,1)$ *so that* $\phi_c = \psi_r \circ \phi_i \circ \psi_{-r}$, *while if* $\Im(c) < 0$, $\exists$ *another* $r \in (-1,1)$, $r \neq 0$ *so that* $\phi_c = \psi_r \circ \phi_{-i} \circ \psi_{-r}$,

Proof: Let $r \in (-1,1)\ :\ r \neq 0$. Then $\psi_r \circ \phi_\imath \circ \psi_{-r} \in \mathbb{P}(1)$, so if z_r is the zero of $\psi_r \circ \phi_i \circ \psi_{-r}$, then $|z - 1/2| = 1/2$. Letting $c_r = 2z_r - 1$, it suffices to show that $\{c_r\ :\ -1 < r < 1\}$ is the upper half semicircle of T.

A careful computation shows that

$$\psi_r \circ \phi_i \circ \psi_{-r}(z) = \frac{(1-r)^2 - ((1-r)^2 - i(1-r^2))z}{(1-r)^2 + i((1-r^2) - (1-r)^2)z}$$

so that

$$z_r = \frac{(1-r)^2 + i(1-r^2)}{2(1+r^2)}$$

Thus

$$c_r = 2z_r - 1 = \frac{-2r + i(1-r^2)}{1+r^2}.$$

One checks that as r goes from -1 to 1 , c_r traces out the required semicircle. A similar computation for ϕ_{-1} yields a c_r that traces out the lower semicircle of T. #

Theorem 3. $\mathbb{C}(\phi_i)\bigcup\mathbb{C}(\phi_{-\imath}) = \mathbb{P}$, *the collection of all parabolic automorphisms.*

Proof: $\mathbb{P} = \bigcup_{w \in T} \mathbb{P}(w)$. Given $w \in T$, choose $\eta \in \mathbb{A}$ so that $\eta(1) = w$. Then we have

$$\eta \circ \mathbb{P}(1) \circ \eta_{-1} = \mathbb{P}(w).$$

Thus each $\psi \in \mathbb{P}(w)$ is conjugate to some $\phi \in \mathbb{P}(1)$, and our previous result shows that ϕ is conjugate to ϕ_i or to ϕ_{-i}. Thus ψ is conjugate to ϕ_i or to ϕ_{-i}. #

We will now examine the case where $c = \pm i$, $\phi_i(z)$ and its inverse $\phi_i = \phi_{-i}$.

Lemma 3. *The zeroes of the iterates of* $\phi_\imath$ *(and those of* $\phi_{-\imath}$*) form a Blaschke sequence.*

Proof: Multiplying each coefficient of ϕ_i by $(1-i)/2$ shows that

$$\phi_i(z) = \frac{1-(1-i)z}{1+i-z}.$$

An easy computation shows that

$$\phi_i \circ \phi_i(z) = \frac{2-(2-i)z}{(2+i)-2z}$$

and by induction we see that the nth iterate is given by

$$(\phi_i)_n\,(z) = \frac{n-(n-i)z}{n+i-nz}.$$

Thus $a_n = n/(n-i)$ is the zero of $(\phi_i)_n$, $|a_n|^2 = n^2/(n^2+1)$. So $\sum(1-|a_n|^2) < \infty$, and (a_n) is a Blaschke sequence. Essentially the same argument shows that the zeroes of $(\phi_{-i})_n$ also form a Blaschke sequence. #

Lemma 4. *Suppose that $\phi \in \mathbb{A}$ with $\phi_n(a_n) = 0\ \forall n \Rightarrow (a_n)$ is a Blaschke sequence. Then*

i) If $\psi \in \mathbb{A}$ and $\psi \circ \phi_n(b_n) = 0$, $\forall n$, then (b_n) is a Blaschke sequence.

ii)If $\widetilde{\phi} \in \mathbb{C}(\phi)$ and if $(\widetilde{\phi})_n(c_n) = 0$, $\forall n$, then (c_n) is a Blaschke sequence.

Proof: For i) assume (a_n) is a Blaschke sequence and consider $\psi \circ \phi_n(b_n) = 0$, with $\phi_n(z) = \frac{\lambda_n(z-a_n)}{1-\overline{a_n}z}$. So $b_n = \phi_{-n} \circ \psi_{-1}(0)$. Let $\alpha = \psi_{-1}(0) \in D$, and $|\phi_{-n}(\alpha)| = |\frac{(\alpha+\lambda_n a_n)}{(1+\overline{\lambda_n a_n}\alpha)}|$.

Then

$$(1-|b_n|^2) = 1 - |\frac{\alpha+\lambda_n a_n}{1+\overline{\lambda_n a_n}\alpha}|^2 = \frac{|a_n|^2(|\alpha|^2-1)+(1-|\alpha|^2)}{|1+\overline{\lambda_n a_n}\alpha|^2} \leq \frac{2(1-|a_n|^2)}{1-|\alpha|}$$

It follows from our assumption that the sequence (b_n) is a Blaschke sequence. #

For part ii) by our assumption $\widetilde{\phi} = \eta \circ \phi \circ \eta_{-1}$ and so assuming $(\widetilde{\phi})_n(c_n) = 0$ we have $(\widetilde{\phi})_n \circ \eta(d_n) = 0$, where $\eta(d_n) = c_n$. Thus $\eta \circ \phi_n(d_n) = 0$. By part i) the sequence (d_n) is a Blaschke sequence. #

Theorem 4. *If $\phi \in \mathbb{A}$ is parabolic and $\psi \in \mathbb{A}$, then $M_\psi U_\phi$ is not Crownover.*

Proof: Suppose ϕ is parabolic.Then $\phi \in \mathbb{C}(\phi_\imath)$ or $\mathbb{C}(\phi_{-\imath})$ so by Lemma 3 and Lemma 4 ii) , $\phi_n(c_n) = 0\ \ \forall n \Rightarrow \{c_n\}$ is a Blaschke sequence. Therefore by Lemma 4 i) , $\Psi \circ \phi_n(b_n) = 0\ \ \forall n \Rightarrow \{b_n\}$ is a Blaschke sequence. The result now follows from Theorem 2.

6. Isometries of Codimension Greater than One.

Recall that if S is an isometry on $H^p, (p \neq 2)$ of codimension $d < \infty$, then $S = M_\Psi C_\phi$ for some $\phi \in \mathbb{A}$ and Ψ a d fold Blaschke product. Our key results for codimension 1 isometries carry over easily to this setting. Thus

(i) if $\widetilde{S} \in B(H^p)$, then

$$\widetilde{S} \approx S \Leftrightarrow \widetilde{S} = \rho M_{\Psi\circ\eta} U_{\eta_{-1}\circ\phi\circ\eta}$$

for some $\eta \in \mathbb{A}$ and a $\rho \in T$ which is determined by ϕ and ρ.

(ii) $\bigcap_n^\infty S^n H^p = (0) \Leftrightarrow \phi$ is elliptic.

For (ii) , one observes that the zeros of ($\Psi \circ \phi_n$, $n \geq 0$) can be written as a union of d Blaschke sequences.

We now consider isometries S of infinite codimension. These can arise in two ways. S could have the form $M_F C_\Phi$ where Φ is inner and $\Phi \notin \mathbb{A}$. The other possibility is that $S = M_\Psi U_\phi$ where $\phi \in \mathbb{A}$ and Ψ is inner and not a finite Blaschke product. We focus on this latter case.

Note that $\bigcap_n^\infty S^n H^p = (0)$ if ϕ is elliptic.

Proposition 6. *Suppose $\phi \in \mathbb{A}$ and ϕ is parabolic or hyperbolic. Depending on this choice of Blaschke product Ψ, the isometry*

$$S = M_\Psi U_\phi$$

can satisfy either $\bigcap_1^\infty S^n H^p = (0)$ or $\bigcap_1^\infty S^n H^p \neq (0)$

Proof. The following proof will utilize the results of Theorems 2 and 4. Suppose that $\phi \in \mathbb{A}$ is parabolic and thus by Proposition 1 we choose $\{\lambda_n\}$ in T such that $\Psi = \prod_1^\infty \lambda_n \phi_{-n}$ is a convergent Blaschke product.

Note that

$$\Psi \circ \phi = \prod_1^\infty \lambda_n \phi_{-n+1} = e\, \Psi.$$

Iterating this step , we see have $\Psi \circ \phi_n$ has Ψ as a factor $\forall n > 0$. Thus $\prod_1^\infty \Psi \circ \phi_n$ can not be a convergent Blaschke product. Hence, $\bigcap_1^\infty S^n H^p = \{0\}$ for $S = M_\Psi U_\phi$.

Now suppose that $\phi_n(a_n) = 0\ \forall n > 0$, so that we must have $\sum_1^\infty (1 - |a_n|) = R < \infty$. Choose $1 < n_1 < n_2 < .....$, such that $\forall k \geq 1$,

$$\sum_{n=n_k}^\infty (1 - |a_n|) < R/(2^k)$$

and let $\Psi = \prod_{k=1}^\infty \phi_{n_k}$. Since

$$\sum_{k=1}^\infty \sum_{n=n_k}^\infty (1 - |a_n|) < \sum_{k=1}^\infty R/(2^k) < \infty,$$

we see that the zeroes of $\prod_1^\infty \Psi \circ \psi_n$ form a Blaschke sequence. Thus $\prod_1^\infty \Psi \circ \phi_n$ is convergent, and $\bigcap_1^\infty S^n H^p \neq \{0\}$ for $S = M_\Psi U_\phi$.

7. Commuting Automorphisms

In this section we elaborate on the conclusion on Corollary 2 by describing the automorphisms of D that commute with a fixed automorphism. These results are undoubtedly known and we outline proofs using the results from the last section.

Definition 9. *For $\phi \in \mathbb{A}$, let*

$$Com(\phi) = \{\psi \in \mathbb{A}\ :\ \psi \circ \phi = \phi \circ \psi\}.$$

Clearly $Com(\phi)$ is a subgroup of $\mathbb{A}$, and $Com(\ e\ ) = \mathbb{A}$.

Proposition 7. *Let $\phi \in \mathbb{A}$, $\phi \neq e$.*

i) If $\phi \in \mathbb{E}(a)$, then $Com(\phi) = \mathbb{E}(a) \bigcup \{e\}$.

ii) If $\phi \in \mathbb{P}(w)$, then $Com(\phi) = \mathbb{P}(w) \bigcup \{e\}$.

iii)If $\phi \in \mathbb{H}(w_1 , w_2)$, then $Com(\phi) = \mathbb{H}(w_1 , w_2) \bigcup \{e\}$.

iv) In each of these cases , $Com(\phi)$ is abelian.

Proof: Observe that $Com(\eta \circ \phi \circ \eta_{-1}) = \eta \circ Com(\phi) \circ \eta_{-1}$. So it will suffice to consider $a = 0$, $w = 1$, and $w_1 = -1$, and $w_2 = 1$.

For i), if $\phi \in \mathbb{E}(0) = \{\eta_\lambda(z) = \lambda z \ : \ \lambda \in T , \lambda \neq 1\}$, then $\mathbb{E}(0) \subset Com(\phi)$. But if $\psi \in Com(\phi)$, then $\psi(0) = \psi(\phi(0)) = \phi(\psi(0))$, so $\psi(0) = 0$. Thus $\psi \in \mathbb{E}(0)$ or $\psi = e$.

For ii) , if

$$\phi \in \mathbb{P}(1) = \{\phi_c = \frac{1 + c - 2z}{1 + i - z} \ : c \in T \ , \ c \neq \pm 1\},$$

then a computation shows that $\mathbb{P}(1)$ is an abelian set. The same argument as in i) shows that if $\psi \in Com(\phi)$, then $\psi = e$ or ψ has 1 as it unique fixed point. So $Com(\phi) = \mathbb{P}(1) \bigcup (\{e\})$.

For iii), suppose that

$$\phi \in \mathbb{H}(-1 \ , \ 1) = \{\psi_r(z) = \frac{z - r}{1 + rz} \ : -1 < r < 0, or, \ 0 < r < 1\}$$

Observe that $\mathbb{H}(-1 \ , \ 1)$ is an abelian set. Let $\psi \in Com(\phi)$. It follows easily that

$$\{\psi(1) \ , \psi(-1) \ \} = \{\pm 1\}.$$

If $\psi(1) = 1$ and $\psi(-1) = -1$, then $\psi \in \mathbb{H}(-1 \ , \ 1)$ as desired. So suppose $\psi(-1) = 1$ and $\psi(1) = -1$. Then $-\psi \in \mathbb{H}(-1 \ , \ 1)$ so $\psi(z) = \frac{r_1 - z}{1 - r_1 z}$ for some r_1. A computation will show that ψ does not commute with automorphisms in $\mathbb{H}(-1 \ , \ 1)$. Thus $Com(\phi) = \mathbb{H}(-1 \ , \ 1) \bigcup \{e\}$. #

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Univ. of North Carolina, Chapel Hill, N.C. $27599 - 3250$